Controlling soliton excitations in Heisenberg spin chains through the magic angle

Jing Lu (卢竞),[1,2] Lan Zhou (周兰),[1] Le-Man Kuang (匡乐满),[1] and C. P. Sun (孙昌璞)[2,*]
[1]*Department of Physics, Hunan Normal University, Changsha 410081, China*
[2]*Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing, 100080, China*
(Received 17 February 2008; published 20 January 2009)

We study the nonlinear dynamics of collective excitation in an N-site XXZ quantum spin chain, which is manipulated by an oblique magnetic field. We show that, when the tilted field is applied along the magic angle, $\theta_0 = \pm \arccos\sqrt{1/3}$, the anisotropic Heisenberg spin chain becomes isotropic and thus an freely propagating spin wave is stimulated. Also, in the regime of tilted angles larger and smaller than the magic angle, two types of nonlinear excitations appear: bright and dark solitons.

DOI: 10.1103/PhysRevE.79.016606 PACS number(s): 05.45.Yv, 75.10.Pq, 76.60.Lz

I. INTRODUCTION

A crucial challenge for quantum-information processing (QIP) is perfectly transmitting a quantum state from one place to another as well as storing the information of a quantum state. Recently, as much attention has been paid to solid-based quantum computing, there have been many proposals for quantum-state transfer and storage, using quantum spin systems such as the quantum data bus and quantum memory [1–4].

On the other hand, the Heisenberg chain is a typical spin system in condensed-matter physics. It has attracted considerable attention for a long time in describing various magnetic properties of materials. As a strongly correlated system [5–9], it can display rich nonlinearities induced by interspin interactions. A typical nonlinear phenomenon is the solitary wave, a wave packet propagating without either energy loss and wave packet spreading. In this sense it is quite natural to consider the role of soliton waves in quantum-state transfer and quantum-information storage.

Generally, there are two major approaches to theoretically study the solitary excitation in the one-dimensional (1D) Heisenberg chain [11–19]. (a) Each spin is represented as a Bloch vector, and a solitary excitation is found by investigating the nonlinear dynamics of this classical variable [12,13]. (b) Another procedure is to employ the boson mappings of spin operators via the Holstein-Primakoff transformation and Jordan-Schwinger realization [10]. By making use of the spin-coherent-state representation, a solvable nonlinear differential equation can be deduced from the Heisenberg equation for the mapping boson operators; therefore, a solitary excitation is implied in the system [14–19].

Our present investigation is motivated by the well-known knowledge in nuclear magnetic resonance (NMR) that the effective spin-spin interaction can be controlled by the angle of the external applied magnetic field [10]. Here, to some extent, the interspin coupling can be canceled effectively by the magic angle. Nowadays together with spin-echo technology, magic angle manipulation has became a necessary tool for controlling the interacting spin system to be free of decoherence due to the interspin couplings. In this paper, we apply a magnetic field on an N-site XXZ spin chain, which is rotated around the y axis by an amount of magnitude θ, and study how to manipulate a soliton excitation by this oblique field. It is known that the ferromagnetic order is completely determined by the direction of the external magnetic field. Indeed, our investigation shows that the switch between the bright and dark solitons is controlled by θ, whose tangent is defined as the ratio between the z and x components of the magnetic field, and at the turning magnitude referred to as the magic angle θ_0, the original anisotropic chain becomes isotropic; therefore, only an ideal spin wave is stimulated as an effective plane wave. Here, we employ the second approach mentioned above, but go beyond the spin-wave approach by considering the nonlinear effect of collective spin excitation. Generally, the spin-wave approach is regarded as a mean-field method with a given order parameter. Due to the lack of knowledge of the nonlinear fluctuation, the spin-wave approach does not cover quantum fluctuations in the nonlinear regime. However, our approach overcomes this disadvantage.

The remainder of this paper is organized as follows. In Sec. II, we present our model: a Heisenberg spin chain with anisotropic coupling in an oblique magnetic field. The approach we use is given in Secs. III and IV. Here the quasiclassical equation of motion is obtained, which is a nonlinear Schrödinger (NLS) equation. In Sec. V, the time evolution of nonlinear excitations is studied. The bright and dark solitons can be excited as θ shifted to the left or right of the magic angle. In Sec. VI, we give a remark about the relationship between the soliton-wave propagation and perfect quantum-state transfer.

II. SPIN CHAIN MODEL

We consider a one-dimensional XXZ spin chain with N spins in an homogeneous external magnetic field

$$\mathbf{B} = B(e_x \sin\theta + e_z \cos\theta). \tag{1}$$

Denote the spin operator at the jth site by the Pauli spin operators S_j^x, S_j^y, and S_j^z. Then the Hamiltonian H of this system reads

*suncp@itp.ac.cn

$$H = -J\sum_{j=1}^{N} (\hat{S}_j^x\hat{S}_{j+1}^x + \hat{S}_j^y\hat{S}_{j+1}^y + \Delta\hat{S}_j^z\hat{S}_{j+1}^z) + \mathbf{B}\cdot\mathbf{S}, \tag{2}$$

where $\mathbf{S}=\Sigma_j\mathbf{S}_j$ is the total spin operator. Three types of interactions are included in Eq. (2): (a) the isotropic part of the nearest-neighbor exchange interaction $J_x=J_y=J>0$; (b) the anisotropic part of nearest neighbor exchange interaction, which is characterized by a dimensionless parameter $\Delta=J_z/J$; and (c) the external homogeneous field $\mathbf{B}$, which is specified by its components $B_x=B\sin\theta$ and $B_z=B\cos\theta$. Here the magnitude of the magnetic field $B=\sqrt{B_x^2+B_z^2}$ and the oblique angle $\theta=\arctan(B_z/B_x)$.

Obviously, the last term is proportional to the total spin z component $\hat{S}^z_{\text{total}}=\Sigma_{\mathbf{i}}\hat{S}^z_{\mathbf{i}}$, which is conserved and thus has common eigenstates with the total Hamiltonian. Correspondingly, the Hilbert space of the system can be decomposed into a direct sum of numerous subspaces $V(M)$ specified by the total spin number M along the z axis. It had been proved [20,21] that, on a finite simple cubic lattice, the ground state of the XXZ model is nondegenerate in a subspace $V(M)$. In particular, its global ground state $\Psi_0(\Delta)$ is just the ground state of the model in the subspace $V(M=0)$. When the lattice is finite, the ground-state energy $E_0(\Delta)$ and the spin-correlation function $\langle\hat{S}^z_{\mathbf{i}}\hat{S}^z_{\mathbf{i}}\rangle$ are analytical with respect to Δ.

Here, we assume that the external magnetic field is much larger than the interspin interaction. Therefore we switch to a new "reference frame" with new Z axis along the quantized direction of spin along the effective field $\mathbf{B}$. The corresponding $\mathbf{S}_j\rightarrow\mathbf{L}_j$ transformation for each spin at site j reads

$$L_j^x = \hat{S}^x_{\mathbf{j}}\cos\theta - \hat{S}^z_{\mathbf{j}}\sin\theta,$$

$$L_j^y = \hat{S}^y_{\mathbf{j}},$$

$$L_j^z = \hat{S}_j^z\cos\theta + \hat{S}_j^x\sin\theta. \tag{3}$$

Therefore the Hamiltonian in Eq. (2) is decomposed into a direct sum of the SO(3) irreducible tensors with respect to $\mathbf{L}_j$—i.e.,

$$H = BL_z + \sum_{M=-2}^{M=2} H_M, \tag{4}$$

which is written according to the irreducible representation $D^{[L]}(L=2,1,0)$ of the SO(3) group. Here, $L_z=\Sigma_j L_j^z$ is the third component of the total angular momentum. With $\delta=(\Delta-1)$,

$$H_0 = -\left(1+\frac{\delta}{2}\sin^2\theta\right)J\sum_j \mathbf{L}_j\cdot\mathbf{L}_{j+1} - \frac{\delta}{2}J(3\cos^2\theta-1)\sum_j L_j^zL_{j+1}^z \tag{5}$$

belongs to the rank-0 representation $D^{[0]}$ of SO(3), while

$$H_{\pm1} = \frac{\delta}{4}J\sin 2\theta\sum_j (L_j^zL_{j+1}^{\pm} + L_j^{\pm}L_{j+1}^z) \tag{6}$$

and

$$H_{\pm2} = -\frac{\delta}{4}J\sin^2\theta\sum_j L_j^{\pm}L_{j+1}^{\pm} \tag{7}$$

belong to the rank-1 and -2 representations $D^{[1]}$ and $D^{[2]}$, respectively. The Mth-order tensors H_M satisfy

$$[L_z,H_M] = MH_M, [L_z,H_0] = 0. \tag{8}$$

From the point of view of perturbation theory, the eigenvalues of the Hamiltonian are those of BL_z at zeroth order—i.e. $L_z=M$—which means that the ground state has all spins parallel to the field and is not degenerate. However, the excited state, where one or two spins are flipped with respect to the ground state, is degenerate. Therefore, when perturbation theory is applied, we diagonalize the Hamiltonian H in each subspace of a given M, which means that at lowest order, the Hamiltonian H_0 is kept. Thus, H_0 plays an indispensable role in governing the dynamics of the spin chain. It can be found directly from the Hamiltonian H_0 that the spin-spin interaction is controlled by the direction of the applied magnetic field. When $\theta=\theta_0=\pm\arccos\sqrt{1/3}$, the original anisotropic chain becomes isotropic.

We further explain the above argument from the point view of the representation theory of SO(3) together with the rotating-wave approximation. Sandwiched by the common states $|J,M'\rangle$ and $|J,M''\rangle$ of H_0 and L_z, Eqs. (8) lead to

$$\begin{aligned}\langle J,M'|[L_z,H_M]|J,M''\rangle &= (M'-M'')\langle J,M'|H_M|J,M''\rangle\\ &= M\langle J,M'|H_M|J,M''\rangle.\end{aligned} \tag{9}$$

Here $M=M'-M''$ for the nonvanishing matrix element $\langle J,M'|H_M|J,M''\rangle$. In the interaction picture, the off-diagonal elements $\langle J,M'|H_M|J,M''\rangle$ are fast changing with high frequency; therefore, terms with $|M|=1$ and $|M|=2$ in the total Hamiltonian can be ignored [10].

III. QUASICLASSICAL MOTION EQUATION FOR FERROMAGNETIC SPIN CHAINS

With the above considerations, $H_M(M=\pm1,\pm2)$ are first- and second-order tensor operators that transform according to the representation $D^{[j=1,2]}$ of the SO(3) group. In the large external field limit, the Hamiltonian (4) is reduced to

$$H = BL_z + H_0. \tag{10}$$

In this section, we begin with the above Hamiltonian to discuss the soliton excitations.

To describe the spin excitation, we introduce the boson excitation by the Holstein-Primakoff transformation [22],

$$L_i^+ = \hat{a}_j^\dagger\sqrt{2S-\hat{a}_j^\dagger\hat{a}_j}, \tag{11a}$$

$$L_i^- = \sqrt{2S-\hat{a}_j^\dagger\hat{a}_j}\,\hat{a}_j, \tag{11b}$$

$$L_i^z = \hat{a}_j^\dagger \hat{a}_j - S, \tag{11c}$$

where the annihilation operators $\hat{a}_i$ and the creation operators $\hat{a}_j^\dagger$ satisfy the boson commutation relation $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$. The number operator $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$ characterizes the spin deviation from its maximum value S of L_i^z. Usually, for a Heisenberg ferromagnetic system with interspin couplings $-J\mathbf{L}_i \cdot \mathbf{L}_{i+1}$ ($J > 0$), spontaneous symmetry breaking will happen along the direction of the external magnetic field (say, along the z axis), which is subsequently assumed to approach zero. The ground state implies a ferromagnetic order with all spins along the $+z$ or $-z$ direction. Therefore the magnetization comes into being, which is defined as the nonvanishing average of $L^z = \Sigma_i L_i^z$. The governing equation for nonlinear excitations on ground states can be obtained by expanding $\sqrt{2S - \hat{n}_i} \simeq \sqrt{2S}[1 - \hat{n}_i/(4S)]$ to a series over the low excitation $\langle \hat{n}_i \rangle \ll 2S$. We first expand (11a) and (11b), and keep the terms in first order of $\langle \hat{n}_i \rangle$:

$$L_i^+ \approx \sqrt{2S}\hat{a}_j^\dagger \left(1 - \frac{1}{4S}\hat{a}_j^\dagger \hat{a}_j\right), \tag{12a}$$

$$L_i^- \approx \sqrt{2S}\left(1 - \frac{1}{4S}\hat{a}_j^\dagger \hat{a}_j\right)\hat{a}_j. \tag{12b}$$

Consequently, beyond the spin-wave approximation, the low-energy effective Hamiltonian (10) is achieved as

$$\begin{aligned} H = &- c_0 S \sum_j (\hat{a}_j^\dagger \hat{a}_{j+1} + \text{H.c.}) + B \sum_j n_j \\ &+ \frac{c_0}{4} \sum_j (\hat{a}_j^\dagger n_j \hat{a}_{j+1} + n_j \hat{a}_j \hat{a}_{j+1}^\dagger + \text{H.c.}) \\ &- (c_0 + c_1) \sum_j n_j (n_{j+1} - 2S), \end{aligned} \tag{13}$$

where $n_j = \hat{a}_j^\dagger \hat{a}_j$, $c_0 = J(1 + \delta \sin^2 \theta/2)$ and $c_1 = \delta J(3\cos^2\theta - 1)/2$.

As we should emphasize that the external magnetic field has been assumed $B > 0$ in the above discussions. If $B < 0$, another form of the Holstein-Primakoff transformation should be taken,

$$L_i^+ = \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}\,\hat{a}_j, \tag{14a}$$

$$L_i^- = \hat{a}_j^\dagger \sqrt{2S - \hat{a}_j^\dagger \hat{a}_j}, \tag{14b}$$

$$L_i^z = S - \hat{a}_j^\dagger \hat{a}_j, \tag{14c}$$

but the main result of this paper is the same. So without loss of generality, $B > 0$ is assumed in the rest of this paper.

The Hamiltonian (13) characterizes the low-energy nonlinear property of ferromagnetic spin chains in an oblique magnetic field. The corresponding Heisenberg equation

$$\begin{aligned} i\hbar \frac{d\hat{a}_j}{dt} = &- c_0 S(\hat{a}_{j+1} + \hat{a}_{j-1}) + (2S(c_0 + c_1) + B)\hat{a}_j \\ &+ \frac{c_0}{4}(2n_j \hat{a}_{j\pm 1} + n_{j\pm 1}\hat{a}_{j\pm 1} + \hat{a}_{j\pm 1}^\dagger \hat{a}_j^2) \\ &- (c_0 + c_1)\hat{a}_j n_{j\pm 1} \end{aligned} \tag{15}$$

contains various nonlinear couplings. They lead to different nonlinear "phases," which are represented by various types of soliton excitations.

IV. CONTINUUM FIELD APPROACH FOR NONLINEAR COLLECTIVE EXCITATION

In this section, we make use of the continuum field theory to study the nonlinear excitations in the ferromagnetic spin chain. To this end we consider the spin-wave approach with some nonlinear corrections. First let us introduce the p representation (also called the Glauber coherent-state representation [23]) defined by the product of the multimode coherent states $|\alpha\rangle = \Pi_i |\alpha_i\rangle$, where each component $|\alpha_i\rangle$ is an eigenstate of the annihilation operator $\hat{a}_i$—i.e., $\hat{a}_i|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$—and α_i is the coherent amplitude. Since coherent states are normalized and overcompleted, the field operator sandwiched by $|\alpha\rangle$ can be represented only with their diagonal elements. Thus, we only need to consider the diagonal parts of Eq. (15), which are enough to describe the nonlinear dynamics without any help of off-diagonal elements. The p representation of the nonlinear equations (15) reads

$$\begin{aligned} i\hbar \frac{d\alpha_j}{dt} = &- c_0 S(\alpha_{j\pm 1} - 2\alpha_j) + (2Sc_1 + B)\alpha_j \\ &+ \frac{c_0}{4}(2|\alpha_j|^2 \alpha_{j\pm 1} + |\alpha_{j\pm 1}|^2 \alpha_{j\pm 1} + \alpha_{j\pm 1}^* \alpha_j \alpha_j) \\ &- (c_0 + c_1)\alpha_j |\alpha_{j\pm 1}|^2. \end{aligned} \tag{16}$$

From then on, the spin dynamics is expressed in terms of the c-number equation in Glauber's coherent-state representation. However, Eq. (16) is difficult to solve due to its nonlinearity and discreteness. Since we are looking for excitations with a length scale much larger than the lattice constant, we take the long-wave approximation; that is, the continuum field theory approach will be employed in the large-N limit. By assuming that the coherent amplitude is continuum in space, the discrete variables $\alpha_i(t)$ can be replaced by a mean field $\varphi(z,t)$—i.e., $\alpha_i(t) \to \varphi(z,t)$. Correspondingly, the difference becomes the differential

$$\alpha_{i\pm 1} - \alpha_i \to \pm \frac{\partial}{\partial z}\varphi + \frac{1}{2}\frac{\partial^2}{\partial z^2}\varphi + \cdots, \tag{17}$$

where the lattice constant is assumed to be unity. By keeping the derivation terms to second order $\partial^2/\partial z^2$, the system of equations (16) becomes a field equation with self-coupling terms,

$$i\hbar\frac{d}{dt}\varphi + c_0 S\frac{\partial^2}{\partial x^2}\varphi + 2c_1\varphi|\varphi|^2 = V\varphi, \tag{18}$$

which is a typical NLS equation in a constant potential $V = 2Sc_1 + B$.

To separate the rapid oscillation in the inhomogeneous nonlinear equation (18), we rewrite $\varphi(z,t)$ as

$$\varphi(z,t) = e^{-i\chi t}\phi(\xi,t), \tag{19}$$

where $\xi = x\sqrt{\hbar/(c_0 S)}$, $\chi = (2Sc_1+B)/\hbar$, and $\phi(\xi,t)$ is the slowly varying envelope. Then the standard form of the NLS equation is obtained,

$$i\phi_t + \phi_{\xi\xi} + 2\frac{c_1}{\hbar}|\phi|^2\phi = 0, \tag{20}$$

which has the soliton solution. Here, the strength of the nonlinear term

$$c_1 = \frac{\delta}{2}J(3\cos^2\theta - 1) \tag{21}$$

is determined by the spin-spin interaction parameter J and the angle θ of the oblique magnetic field. We notice that when $\theta = \theta_0 = \arccos\sqrt{1/3}$, the nonlinear term in the equation of motion disappears; therefore, Eq. (20) becomes a standard wave equation, which gives a complete set of plane-wave solutions. In this case, the collective excitations are spin waves. θ_0 is named the magic angle. It shows that the magic angle changes the anisotropic system to an isotropic one.

V. NONLINEAR TIME EVOLUTION OF LOCALIZED MAGNETIZATION

For this ferromagnetic spin chain, the interaction parameter J is positive. Thus the sign of coefficient c_1 is determined by the angle of the oblique magnetic field θ. Also, the nonlinear property of this system is strongly related to the sign of the nonlinear term. With these different physically accessible parameters, the nonlinear equation (20) can possess bright- and dark-soliton solutions. Next, we use the inverse scattering method [24,25] to obtain different solitons of this system.

When $c_1 > 0$—that is, $\delta(3\cos^2\theta - 1) > 0$—the single bright-soliton solution of Eq. (18) is obtained as

$$\varphi(x,t) = Ae^{i(\gamma x - \omega t)}\,\mathrm{sech}\left[A\sqrt{\frac{c_1}{c_0 S}}(x - x_0 - vt)\right], \tag{22}$$

with $v = v_1\sqrt{c_0 S/\hbar}$, $\gamma = \hbar v/2c_0 S$, and

$$\omega = \frac{c_1}{\hbar}(2S - A^2) + \frac{B}{\hbar} + \frac{\hbar v^2}{4c_0 S}, \tag{23}$$

where v_1 is an integral constant. The coefficient $v = v_1\sqrt{c_0 S/\hbar}$ describes the velocity of the bright soliton traveling to the right. The positive coefficient A characterizes the size of the bright soliton. The coefficient x_0 denotes the initial position of the bright soliton. The parameters v_1, A, and x_0 are determined by the initial state. The bright-soliton solution (22) depicts a wave packet traveling in a continuous background.

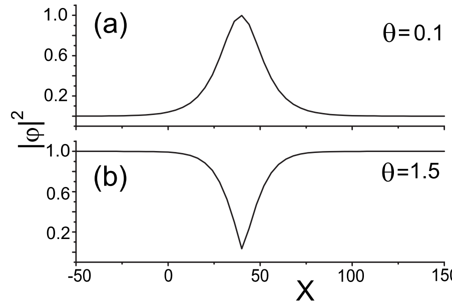

FIG. 1. Bright soltion (a) obtained from Eq. (22) for $\theta = 0.1$. Dark soltion (b) obtained from Eq. (24) for $\theta = 1.5$. Other parameters are taken as follows: $S = 10$, $B = 100$, $J = 1$, $\delta = 0.1$, $x_0 = x_0' = 0$, $A = A' = 1$, $v_1 = v_1' = 5$, and $t = 3$.

When $c_1 < 0$—that is, $\delta(3\cos^2\theta - 1) < 0$—we can get a single dark-soliton solution of Eq. (18),

$$\varphi(x,t) = A' e^{i(\gamma' x - \omega' t)}\tanh\left[A'\sqrt{\frac{-c_1}{c_0 S}}(x - x_0' - v' t)\right], \tag{24}$$

with $v' = v_1'\sqrt{c_0 S/\hbar}$, $\gamma' = \frac{\hbar v'}{2c_0 S}$, and

$$\omega' = 2\frac{c_1}{\hbar}(S - A'^2) + \frac{B}{\hbar} + \frac{\hbar v'^2}{4c_0 S}, \tag{25}$$

where v_1' is an integral constant. $v' = v_1'\sqrt{c_0 S/\hbar}$ characterizes the velocity of the dark soliton traveling to the right. The positive coefficient A' characterizes the size of the dark soliton. The coefficient x_0' denotes the initial position of the dark soliton. The parameters v', A', and x_0' are decided by the initial state. The dark-soliton solution describes a localized dip in the continuous background. Figure 1 numerically illustrates the bright and dark solitons obtained from Eqs. (22) and (24), respectively, in comparison with the results with many extra nonlinear terms in the following discussion.

Furthermore, to test the rationality for neglecting the terms $H_{\pm 1}$ and $H_{\pm 2}$, we deal with the Hamiltonian (4) by the approach presented above. The Holstein-Primakoff transformation (11) is first employed to write Hamiltonian (4) in terms of bosonic operators:

$$\begin{aligned}
H = &- c_0 S\sum_j (\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_j \hat{a}_{j+1}^\dagger) + B\sum_j n_j - (c_0 + c_1)\\
&\times \sum_j n_j(n_{j+1} - 2S) + \frac{c_0}{4}\sum_j (\hat{a}_j^\dagger n_j \hat{a}_{j+1} + \hat{a}_j^\dagger n_{j+1}\hat{a}_{j+1} + \text{H.c.})\\
&+ c_2\sqrt{2S}\sum_j \left(\hat{a}_j^\dagger n_{j\pm 1} + \frac{1}{2}\hat{a}_j^\dagger n_j - 2S\hat{a}_j^\dagger + \text{H.c.}\right)\\
&- 2c_3 S\sum_j \left[-\frac{1}{4S}(\hat{a}_j^\dagger n_j \hat{a}_{j+1}^\dagger + \hat{a}_j^\dagger \hat{a}_{j+1}^\dagger n_{j+1}) + \hat{a}_j^\dagger \hat{a}_{j+1}^\dagger + \text{H.c.}\right],
\end{aligned} \tag{26}$$

where $c_0 = J(1 + \delta\sin^2\theta/2)$, $c_1 = J\delta(3\cos^2\theta - 1)/2$, $c_2 = J\delta\sin 2\theta/4$, and $c_3 = J\delta\sin^2\theta/4$. The Hamiltonian (26) characterizes the low-energy nonlinear property of ferromagnetic spin chains in an oblique magnetic field. The corresponding Heisenberg equation

$$i\hbar\frac{d\hat{a}_j}{dt} = - c_0S\hat{a}_{j\pm1} + [B + 2S(c_0 + c_1)]\hat{a}_j - (c_0 + c_1)\hat{a}_jn_{j\pm1} + \frac{c_0}{4}(2n_j\hat{a}_{j\pm1} + n_{j\pm1}\hat{a}_{j\pm1} + \hat{a}^\dagger_{j\pm1}\hat{a}_j\hat{a}_j) + c_2\sqrt{2S}\left(n_{j\pm1} + \hat{a}^\dagger_{j\pm1}\hat{a}_j + \hat{a}_{j\pm1}\hat{a}_j - 2S + n_j + \frac{1}{2}\hat{a}_j\hat{a}_j\right) - c_3 2S\left(\hat{a}^\dagger_{j\pm1} - \frac{1}{2S}n_j\hat{a}^\dagger_{j\pm1} - \frac{1}{4S}\hat{a}^\dagger_{j\pm1}n_{j\pm1} - \frac{1}{4S}\hat{a}_j\hat{a}_j\hat{a}_{j\pm1}\right) \quad (27)$$

contains rich nonlinear couplings and thus can predict various nonlinear phenomena.

Further, we use the continuum field theory approach—that is, expressing the spin dynamics according to its corresponding continuous c-number equation in Glauber's coherent-state representation. Then a similar field equation is obtained as

$$i\hbar\frac{d}{dt}\varphi + c_0S\frac{\partial^2}{\partial x^2}\varphi + 2c_1\varphi|\varphi|^2 = (2Sc_1 + B)\varphi + 2c_2\sqrt{2S}\left(\frac{5}{2}|\varphi|^2 + \frac{5}{4}\varphi^2 - S\right) - c_3(4S\varphi^* - 3\varphi^*|\varphi|^2 - \varphi^3). \quad (28)$$

Obviously, the effects on $H_{\pm1}$ and $H_{\pm2}$ are kept in the above derivation. Equation (28) is similar to the NLS equation, but more nonlinear terms are involved; therefore, an analytic solution cannot be obtained.

To display the effects of extra nonlinear terms, we numerically investigate the time evolution of dark and bright solitons under the action of the differential equation (28) in Figs. 2 and 3. Figures 2 and 3 show that a bright and a dark solition can be excited when θ takes value on the left and right sides of the magic angle, respectively. Comparing the numerical results shown in Figs. 2 and 3, it is found that the bright-soliton is more stable than the dark-soliton solution. This is because the long-time evolution of the dark soliton in Fig. 3 becomes a profile with spatial oscillation. While the bright soliton will almost keep its shape, the dark soliton will disappear after a long enough evolution. Thus the bright soliton is more easily detectable in practice. From Fig. 2, we found that the closer to the magic angle the tilted angle is, the wider the soliton becomes.

In Fig. 4, we plot the time evolution for the stable bright soliton with respect to different values of the dimensionless parameter $\lambda = B/J$. It can be found that the extra nonlinear terms have no effect on solitons when $\lambda = 100$. Such a numerical analysis confirms that the solitonlike wave can be excited under an appropriate regime from $\lambda = 10$ to 1000 approximately. Therefore in this regime, it is reasonable to discard the terms $H_{\pm1}$ and $H_{\pm2}$ in Eq. (4). When the parameter λ goes to infinity (see the dotted line, $\lambda = 5000$, in Fig. 4), the interaction term H_0 can be completely ignored; then, the Hamiltonian of the system is approximately described by BL_z. In this situation, the system is completely polarized along the direction of the magnetic field and forms a background with symmetry breaking which is displayed by the dotted line in Fig. 4. The numerical result—that a very sharp peak is localized around $x=0$—justifies our observation from physical intuition.

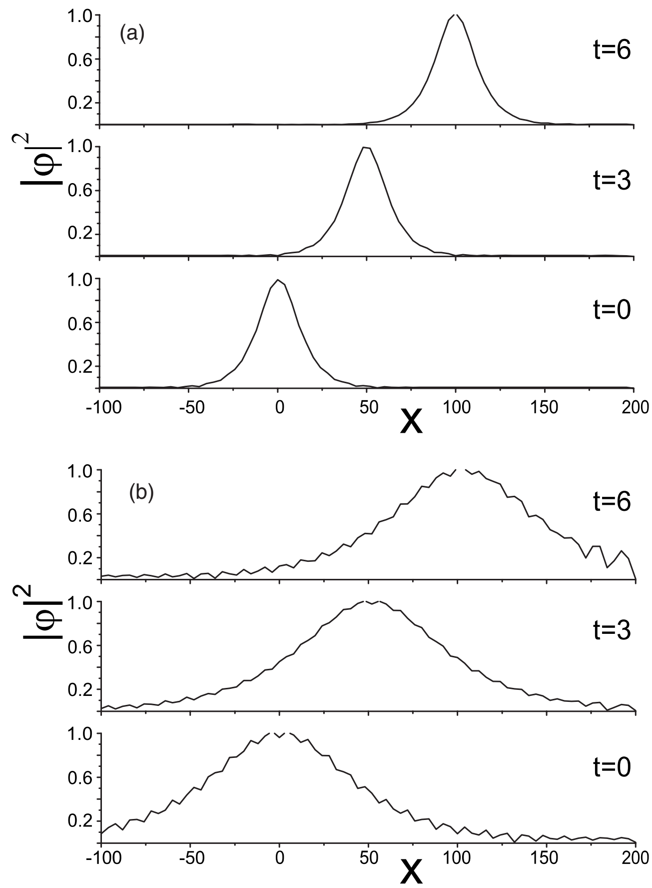

FIG. 2. Numerical time evolution of a bright soliton under the action of Eq. (28) for $\theta = 0.1$ (a) and $\theta = 0.9$ (b). Other parameters are taken as follows: $S=10$, $B=100$, $J=1$, and $\delta = 0.1$. It indicates that a bright soliton is excited when $\theta < \theta_0$.

There are some experimental data on quasi-one-dimensional ferromagnetic chains [27–31], which may be used to test our predictions in an indirect way. For a spin chain made of the material $Ca_3Co_{2-x}Fe_xO_6$ ($x=0,0.1,0.2$) [27], the exchange constant J decreases from 4.39 to 8.13 K and the magnetic field strength $B < 10$ T (~ 33.6 K). Solitary excitation is possible since the dimensionless parameter $\lambda = B/J \approx 10$. However, for the three ferromagnetic chains [28–31], which are made of the material (a) $[(CH_3)_3NH]FeCl_3 \cdot 2H_2O$ [28,29], where $J \sim 17.4$ K and $B < 16$ T (~ 21.5 K); (b) $CoCl_2 \cdot 2D_2O$ [30], where $J \sim 2.475$ K and $B < 6$ T (~ 12.1 K); (c) $CoCl_2 \cdot H_2O$ [31], $J \sim 18.3$ K and $B < 6$ T (~ 12.1 K), the bright soliton may be excited, but noises exist as shown in Fig. 4, with the dimensionless parameter $\lambda = B/J \approx 1$.

VI. REMARKS AND CONCLUSION

Before concluding this paper, let us discuss the physical meaning of the solitary wave of magnetic excitation in an

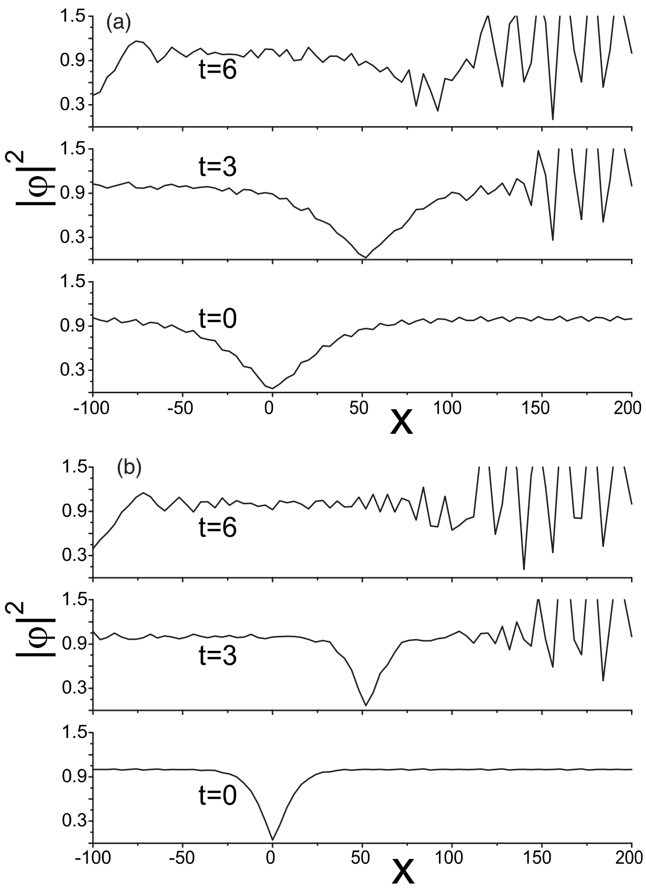

FIG. 3. Numerical time evolution of a dark soliton by the influence of extra nonlinear terms in Eq. (28) for $\theta=1.0$ (a) and $\theta=1.5$ (b). Other parameters are taken as follows: $S=10$, $B=100$, $J=1$, and $\delta=0.1$. It indicates that a dark soliton is excited when θ takes value on the right side of the magic angle.

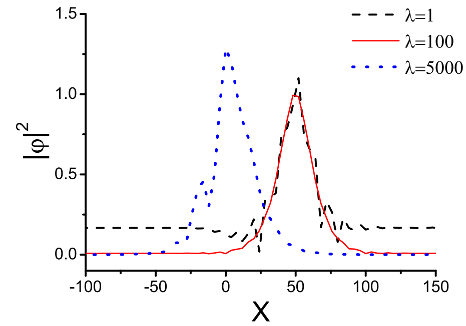

FIG. 4. (Color online) Time evolution of a bright soliton under the effluence of extra nonlinear terms in Eq. (28) with respect to different values of the dimensionless parameter $\lambda=B/J$ at $\theta=0.1$, $S=10$, $\delta=0.1$, and $t=3$.

N-site XXZ quantum spin chain from the point view of quantum-information processing. We first note that the multimode coherent states $|\alpha\rangle=\Pi_j|\alpha_j\rangle$ represent an inhomogeneous collective excitations distributed around the spin chain. This collective excitation has a spin representation by

$$\hat{a}_j^{\dagger}=L_i^{+}\frac{1}{\sqrt{S-L_i^z}}\equiv QL_i^{+}, \tag{29}$$

with $Q=1/\sqrt{S-L_i^z+1}$ and $|0\rangle_j=|S,-S\rangle_j$ being the lowest eigenstate of the on-site spin $\mathbf{L}_j$. We also notice that the coherent state denotes a superposition of various spin states $|S,m_s\rangle_j$—i.e.,

$$|\alpha_i\rangle=\sum C_n(\alpha_i)(QL_i^{+})^n|S,-S\rangle_j=\sum B_n(\alpha_i)|S,n-S\rangle_j, \tag{30}$$

where $C_n(\alpha_i)=\exp(-|\alpha_j|^2/2)\alpha_j^{n_j}/(n_j!)$ and $B_n(\alpha_i)$ can be obtained from $C_n(\alpha_i)$ directly. During the propagation of a soliton, the narrow traveling wave packet does not spread; therefore, it behaves like a "flying qudit" (a $d=2S$ level system).

When $S=1/2$, the qubit is localized at the jth site with a superposition state

$$|\alpha_j\rangle\sim\left|\frac{1}{2},-\frac{1}{2}\right\rangle+\frac{\alpha_j}{2}\left|\frac{1}{2},\frac{1}{2}\right\rangle, \tag{31}$$

which can be used to encode quantum information as usual. During the propagation, the wave function nearly keeps its spatial shape all the time. From an mathematical point of view, the spatially non spreading properties of the carrying excitation wave are very crucial for quantum-state transfer from one location to another with high fidelity. It seems the bosonic excitations obey the bosonic commutation relations only in the large-S limit, but the spin-wave approach can still work well for $S=1/2$ in condensed matter physics. In this sense we can suppose our above arguments valid.

In summary, we have studied solitary magnetic excitation in an N-site XXZ quantum spin chain as well as how to use an oblique magnetic field to create different types of solitons. Through a mean-field approximation beyond the usual spin-wave approach, we obtain the quasiclassical motion equations for nonlinear evolution of the Heisenberg spin system. We show that the switch between the bright and dark solitons is controlled by the angle of the magnetic field, whose tangent is defined by the ratio between the z and x components of the magnetic field. At the magic angle, the system becomes the isotropic Heisenberg model; hence, only an ideal spin wave is stimulated. We also remark on the possibility for solitons to play the role of "flying qudit" based on the well-known results in Ref. [26]—that a no-spreading wave packet behaves like a flying qubit.

ACKNOWLEDGMENTS

This work was supported by the NSFC under Grant Nos. 90203018, 10474104, 60433050, 10325523, 10347128, 10075018, 10775048, and 10704023, the NFRPC with under Grant Nos. 2007CB925204 and 2005CB724508, and the Scientific Research Fund of Hunan Provincial Education Department of China (Grant No. 07C579).

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